

**Seminar in Summer Semester 2024:
Amenable Groups**

Symmetry, the invariance of an object under a group of mathematical transformations, is a key concept in modern mathematics. For finite symmetry groups there is a wonderfully simple trick for creating a symmetrical object from non-symmetrical one: averaging. For example, given an arbitrary function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, we can construct the symmetric function

$$g(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

Supplementing algebra with some analysis, we can push averaging tricks beyond the case of finite groups. This leads to class of amenable groups, which includes all finite groups, all abelian groups, and many non-abelian groups. However, not all groups are amenable. The most famous non-examples are the free groups \mathbb{F}_n for $n \geq 2$. The non-amenability of \mathbb{F}_2 is central to paradoxical constructions such as the volume doubling Banach–Tarski paradox. The Day–von Neumann conjecture asks whether all non-amenable groups contain a copy of \mathbb{F}_2 . Counterexamples were found in the 1980s, but the amenability of the Thompson group remains open.

Suggested Topics

1. Means and finitely additive measures.
2. Amenable groups (via means).
3. Non-amenable groups (esp. free groups).
4. Permanence properties of amenability.
5. Elementary amenable groups.
6. Growth of groups (polynomial, exponential, intermediate).
7. Visualising amenability: Følner sequences.
8. From means to Følner sequences and back again.
9. Representation theory and amenable groups
10. Fixed point theorems for amenable groups
11. Introduction to the Day–von Neumann conjecture
12. Introduction to Thompson’s groups
13. Introduction to the Banach–Tarski paradox.

Practical Info

- The seminar takes place Time TBC Room TBC. Talks may be in English or German.
- The seminar is aimed at bachelor students who have taken courses in functional analysis and group theory.
- The preliminary meeting takes place on **20th March, 2024 at 14:00 (sharp = s.t.) in Room TBC**.
- Students may request preferred topics at the meeting or following the meeting by emailing Dr. Evington.
- You will be expected to submit a draft of the write up at least two weeks before the presentation, at which point a meeting with either Dr. Evington or Prof. Winter should be arranged to discuss the presentation and clarify any details from the material as needed. The final write up should be submitted within a week of the presentation. Naturally we are available for questions outside of this meeting.