Amenable Groups

Seminar: Summer Semester 2024

Prof. Dr. Wilhelm Winter and AR Dr. Samuel Evington

Talks are assigned on a first-come-first-serve basis. Contact one of the organisers in person or by email to request a particular talk.

You should arrange a meeting with one or both of the organisers to discuss a general outline as well as the treatment of supplementary/peripheral material before you begin preparing your lecture. A draft of the lecture notes should be submitted at least two weeks before the presentation, at which point a meeting with one or both of the organisers should be arranged to discuss the presentation and clarify any details from the material as needed. Written lecture notes should accompany any presentation that is to be graded.

Talks

1. Means and finitely additive measures.
Date: 10.04.24
Speaker:
Contents: Briefly recall Banach spaces, their duals, weak* topology, Banach–Alaoglu Theorem. Example: General $\ell^\infty(X)$ for a set X (not just $X = \mathbb{N}$). Elements of the dual space $\ell^\infty(X)^*$, Banach limits, means, finitely additive measures.
Sources: [5, Chapter 4.1-4.2], [8, Chapter 5]

2. Amenable groups (via means)
Date: 17.04.24
Speaker:
Contents: Group actions $G \curvearrowright X$. Induced action on $\ell^\infty(X)$. Invariant means. Define amenability for (discrete) groups. Proof that finite groups are amenable. Proof that $\mathbb{Z}$ is amenable.
Sources: [5, Chapter 4.3-4.4], [8, Chapter 6], [11, Lecture 2] [16], [9]

3. Non-amenable groups (esp. Free groups)
Date: 24.04.24
Speaker:
Contents: Free groups: Definition and Cayley graphs. Paradoxical decompositions. Proof that $\mathbb{F}_n$ is non-amenable. Free products. When is $\mathbb{Z}_n * \mathbb{Z}_m$ amenable?
Sources: [5, Theorem 4.4.7 and Chapter 4.8], [1, Chapter II.5]

May day: no seminar
Date: 01.05.24

4. Permanence properties of amenability
Date: 08.05.24
Speaker:
Contents: Proof that amenability of passes to subgroups, quotients, and extensions. Proof that amenability is preserved by products and increasing unions. Proof that abelian groups are amenable (starting from $\mathbb{Z}$). Sources: [5, Chapter 4.5], [7, VII.2 Amenability], [11, Lecture 2]

5. Elementary amenable groups.
Date: 15.05.24
Speaker:
Contents: Define the class of elementary amenable groups. Concrete examples of non-commutative amenable groups e.g. solvable groups, nilpotent groups with examples. An informal introduction to ordinals and transfinite induction. Are all amenable groups elementary amenable? (Set the stage for talk 6)
Sources: [6], [11, Lectures 3], [11, Lectures 5 or 8]

Pfingstenpause: no seminar
Date: 22.05.24

6. Growth of groups (polynomial, exponential, intermediate)
Date: 29.05.24
Speaker:
Contents: Definition of growth of a (finitely generated) group. Growth of $\mathbb{Z}$ and $\mathbb{Z}^n$ and $\mathbb{F}_n$. Polynomial, exponential, intermediate growth. What groups have polynomial growth? Relationship between growth rate and amenability. Discuss the existence of amenable groups of intermediate growth.
Sources: [5, Chapter 6.4],[10]

7. Visualising amenability: Følner sequences.
Date: 05.06.24
Speaker:
Contents: Definition of Følner sequences. Equivalent formulation of the Følner condition. Examples in $\mathbb{Z}$ and $\mathbb{Z}^n$. Reprove other properties of amenability using Følner sequences.
Sources: [5, Chapter 4.7], [11, Lecture 2]

8. From means to Følner sequences and back again
Date: 12.06.24
Speaker:
Contents: Present the proof that the existence of a left-invariant mean is equivalent to the existence of a Følner sequence/net.
Sources: [5, Theorem 4.9.1],[11, Lecture 2]

9. Representation theory and amenable groups
Date: 19.06.24
Speaker:
Contents: Intro to representation theory. Left regular representation. Weak containment. Amenability as weak containment of the trivial representation. Existence of approximately fixed vectors.
Sources: [12, Chapter 4.2], [11, Lecture 12], [2, Theorem 2.6.8 (1)$\Leftrightarrow$(4)]

10. Fixed point theorems for amenable groups
Date: 26.06.24
Speaker:
Contents: Explain and prove the following theorem: A discrete group G is amenable if and only if every continuous affine action of G on a compact and convex subset of a locally compact vector space has a fixed point. Define any unknown words or concepts in the proof.
Sources: [11, Lecture 13], [5, Theorem 4.10.1], [14]

11. Introduction to the Day–von Neumann conjecture
Date: 10.07.24
Speaker:
Contents: Proof that a (discrete) group containing $\mathbb{F}_2$ is not amenable. The Day–von Neumann conjecture is that the converse holds. Discuss the history of this conjecture and the counterexample in as much detail as you can.
Sources: [5, Chapter 4.2 and 4.5], [9],[13].

12. Introduction to Thompson's groups
Date: 17.07.24
Speaker:
Contents: Define Thompson groups $F \subset T \subset V$, discuss the history, and learn as much as you can about them. Can you explain why $\mathbb{F}_2$ does not embed into F? You are not expected to prove that F is amenable/non-amenable!
Sources: [4],[3].

13. Introduction to the Banach–Tarski paradox
Date: 24.07.24
Speaker:
Contents: What is volume? What is the statement of the Banach–Tarski paradox? Why is our intuition wrong? What is the Axiom of choice and what is its role in the construction? Sketch the construction is as much details as you can. What is the relationship with amenability?
Sources: [8, Chapter 6],[11, Lecture 1], [15]

All sources should be available electronically and/or through the library. If you cannot access some of the source material, please let us know.

References

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[2] P. Brown Nathanial and O. Narutaka. C*-algebras and finite-dimensional approximations. *Graduate Studies in Mathematics*, 88, 2008.

[3] J. Burillo. Introduction to thompson's group f. `https://web.mat.upc.edu/pep.burillo/book_en.php`, 2024.

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[5] T. Ceccherini-Silberstein, M. Coornaert, T. Ceccherini-Silberstein, and M. Coornaert. *Cellular automata*. Springer, 2010.

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[7] K. Davidson. *C*-algebras by example*, volume 6. American Mathematical Soc., 1996.

[8] O. Deiser. *Reelle Zahlen*. Springer-Verlag Berlin Heidelberg, 2007. `www.aleph1.info/?call=Puc&permalink=reellezahlen`.

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[10] R. Grigorchuk and I. Pak. Groups of intermediate growth: an introduction for beginners. arXiv:math/0607384.

[11] K. Juschenko. Lecture notes on amenability. `https://web.ma.utexas.edu/users/juschenko/notes.html`.

[12] D. Kerr and H. Li. *Ergodic theory*. Springer, 2016.

[13] A. Y. Ol'shanskii. On the problem of the existence of an invariant mean on a group. *Russian Mathematical Surveys*, 35(4):180, 1980.

[14] W. Rudin. Functional analysis, mcgrawhill. *Inc, New York*, 45(46):4, 1991.

[15] T. Tao. The banack–tarski paradox. https://www.math.ucla.edu/ tao/preprints/Expository/banach-tarski.pdf.

[16] J. von Neumann. Zur allgemeinen Theorie des Masses. *Fundamenta Mathematicae*, 13:73–116, 1929.